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Mach's Principle and Newtonian Mechanics†

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The union of Mach's principle and Newtonian mechanics gives rise to Relational Mechanics. We find that the characteristics of the revised mechanics are: (1) freedom from any reference to absolute space; (2) the identity of inertial and gravitational mass; (3) the relative acceleration of a body in a gravitational field dependent on the mass of the body. All these results are valid in the context of a Newtonian mechanics which is being developed in the center-of-mass system of all the particles. The conservation of linear momentum, energy, angular momentum are expressed in relational terms, i.e., no reference is made to absolute space. Relational Mechanics is a classical relativistic theory which can be formulated to satisfy Einsteinian relativistic requirements. The Hamiltonian formalism for Relational Mechanics is discussed.

1. *Introduction*

Although the arguments in support of the relativity of motion and position of a body are well known and widely accepted, Newton's laws of motion are still presented (classically and relativistically) with space as the referent for that motion and position. It is our purpose to remove the discrepancy between theory and practice by formulating the laws of motion so that Mach's principle is satisfied. The thesis of that principle is that the position and motion of an object are discernible only in relation to other bodies. In essence, the principle substitutes physical objects for space as the physically meaningful referents for the description of position and motion.

We will carry out such a program for Newtonian mechanics and establish a number of results of value to classical and quantum mechanics. Among these are: (1) A relational form for Newton's second law of motion which eliminates any reference to absolute space. (2) Coordinate systems which are physically determined and for which the concept of inertia is clearly defined. (3) A Hamiltonian formalism, of particular significance for quantum mechanics, expressed in terms of relative coordinates and velocities.

2. *Relational Form for Newton's Second Law of Motion*

For a number of reasons, it will be preferable to discuss Newton's laws of motion as they are applied to a system of N particles which are in the

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gravitational fields of each other. Certain results which will be obtained for this particular application are of interest in themselves and the procedure is easily generalized.

In an obvious notation, in which the reference system is absolute space, we find Newton's equations for the N particles to be

$$m_i \ddot{\mathbf{r}}_i = Gm_i \sum_{k \neq i} m_k(\mathbf{r}_k - \mathbf{r}_i)/|\mathbf{r}_k - \mathbf{r}_i|^3 \tag{2.1}$$

where both i and k run through the integers from 1 to N. The right-hand side of the equation is independent of the space frame of reference, but the left-hand side is not. If we multiply the equation with m_j and subtract from the result the equation obtained by interchanging the subscripts i and j, we find

$$m_i m_j(\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_j) = Gm_i m_j \left\{ \sum_{k \neq i} \frac{m_k(\mathbf{r}_k - \mathbf{r}_i)}{|\mathbf{r}_k - \mathbf{r}_i|^3} - \sum_{k \neq j} \frac{m_k(\mathbf{r}_k - \mathbf{r}_j)}{|\mathbf{r}_k - \mathbf{r}_j|^3} \right\} \tag{2.2}$$

Summing equation (2.1) over all the particles, we obtain

$$\sum_i m_i \ddot{\mathbf{r}}_i = 0 \tag{2.3}$$

which yields the well-known Law of the Conservation of Linear Momentum.

Equations (2.2) and (2.3) are mathematically equivalent to the original set of equation (2.1). However, equation (2.2) is expressed in terms of relative displacements and accelerations, and is, therefore, independent of the choice of coordinate system or frame of reference. Equation (2.3), on the other hand, depends on quantities which require the reference frame of absolute space. Such relations are physically unverifiable and must be eliminated from the theory, or else replaced by a relation which is independent of absolute space. For the moment, we will dispense with equation (2.3) and proceed with the development of the theory based solely on equation (2.2). The results which will be obtained will also be consequences of Newton's original formulation, but we will be unable to reverse our steps and regain equation (2.1). The best that we could achieve would be equation (2.2).

The symmetry of equation (2.2) reflects the independence of the quantities of the frame of reference and the mutual dependence of the particles (bodies) involved. These characteristics are just those required by Mach's principle. Hence, when equation (2.2), or an equivalent form which we will develop below, is taken as the starting point for the development of mechanics, we can be sure that Mach's principle has been imbedded from the start into the theory.

Let us sum equation (2.2) over all the jth particles. We find

$$m_i\{\ddot{\mathbf{r}}_i - (\textstyle\sum m_j \ddot{\mathbf{r}}_j/M)\} = Gm_i \sum_{k \neq i} m_k(\mathbf{r}_k - \mathbf{r}_i)/|\mathbf{r}_k - \mathbf{r}_i|^3; \qquad M = \sum m_j \tag{2.4}$$

or with a slight rearrangement

$$\frac{m_i M(i)}{M}\left(\ddot{\mathbf{r}}_i - \frac{\sum' m_j \ddot{\mathbf{r}}_j}{M(i)}\right) = G m_i \sum_{k \neq i} \frac{m_k(\mathbf{r}_k - \mathbf{r}_i)}{|\mathbf{r}_k - \mathbf{r}_i|^3};$$
$$M(i) = \sum_{j \neq i} m_j, \qquad \textstyle\sum' m_j \ddot{\mathbf{r}}_j = \sum_{j \neq i} m_j \ddot{\mathbf{r}}_j \tag{2.5}$$

Comparison of equation (2.4) with equation (2.1) reveals that except for the reference frame both sets of equations are identical. In Newton's equation (2.1), the acceleration of the ith particle, $\ddot{\mathbf{r}}_i$, is absolute, whereas in equation (2.4) the corresponding acceleration is relative to that of *all* the particles in the system. Note that in this formulation, $\sum m_j \ddot{\mathbf{r}}_j/M$ is the same for all the particles, i.e., for arbitrary ith particle.

Equation (2.5) is a reformulation of equation (2.4) which displays explicitly the acceleration of the ith particle relative to the *remaining* particles in the system. Note that each particle has a different background for the description of its motion. However, equation (2.5) exhibits clearly the dependence of the acceleration of a particle on its mass. In equation (2.4), if we were to divide through with m_i, we would not have completely eliminated m_i from these equations since it occurs in the term $\sum m_j \ddot{\mathbf{r}}_j/M$. Just how the equation of motion depends on m_i is best exhibited by equation (2.5). In conventional terminology, we would describe this equation as a center-of-mass description of the interaction between the ith particle and all the remaining particles of the system. Equation (2.5) yields the result that the acceleration of a particle relative to that of the center-of-mass of the remaining particles depends on the mass of that particle.

The last statement would seem to contradict the oft-stated result that is usually given in Newtonian mechanics. Namely, the acceleration of a body in a gravitational field is independent of the mass of that body. However, the equations which led us to this seemingly contradictory result is a valid deduction from Newton's equations of motion. The resolution of this apparent paradox is to be found in noting that the claim of the independence of the gravitational acceleration on the mass of the body accelerated is based on Newton's equations of motion which describe the motion in absolute space. Indeed, equation (2.1) yields just this result. However, if we refer the motion to the bodies themselves, which is the situation represented by equation (2.4) or (2.5), then we find that the relative acceleration depends on the mass of the particle accelerated.

We can see this explicitly by considering a system of two particles only, i and j. From equation (2.4) or (2.5), we obtain

$$(m_i m_j/m_i + m_j)(\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_j) = G m_i m_j(\mathbf{r}_j - \mathbf{r}_i)/|\mathbf{r}_j - \mathbf{r}_i|^3$$

or

$$\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_j = G(m_i + m_j)(\mathbf{r}_j - \mathbf{r}_i)/|\mathbf{r}_j - \mathbf{r}_i|^3 \tag{2.6}$$

We may identify the i–j particles with the sun–earth, moon–earth, or terrestrial object–earth system. In any case, equation (2.6) is the well-known equation for such systems and it clearly states the dependence of the relative acceleration on the masses of the component parts. Thus, even in the context of Newtonian mechanics, there is no paradox. However, from an observational point of view, we must reject the implications based on absolute space and replace them with those which are founded on the relational aspect of position and motion. Such a viewpoint is given by equation (2.4) or (2.5) in which the fiducial system is either the entire system or else the system without the particular object whose motion is being described. It is this set of equations which is supported by experimental data (Goldstein, 1950). Henceforth, we will refer to equation (2.4) or (2.5) as the relational form for Newton's equation of motion. What we have so far established will be shown to hold in general for arbitrary force fields.

3. *Physically Determined Coordinate Systems*

Equation (2.2), as we have pointed out, is independent of the choice of coordinate system. The same can be said of the relational equations (2.4) and (2.5) which are derived from equation (2.2). But among these coordinate systems are those whose origins are determined by the physical system itself. In equation (2.4), for example, the center-of-mass of the physical system can be chosen as the origin of a particular coordinate system. For this choice, our equations reduce to the classical form for Newton's equations of motion. Another physically determined set of coordinates arises from equation (2.5) when the origin of a coordinate system is taken to be the center-of-mass of all the particles but one—the one excluded is the particle whose behavior is under investigation.

Were it not for the weakness of the gravitational field, the choice of physically determined coordinate systems would be severely restricted. We would, at all times, have to take into account all the particles in the universe. But a wider choice is possible, since, from an observational view-point, we cannot detect, in general, the effects of gravitation on most of the phenomena we observe.

To explore these statements mathematically, let us return to equation (2.4). If we sum over all the particles in the system, we find that

$$\sum_i m_i\{\dot{\mathbf{r}}_i - (\sum m_j \dot{\mathbf{r}}_j/M)\} = 0 \tag{3.1}$$

This result is a mathematical identity. Yet it is the relational equivalent to equation (2.3) and states that the sum of the relative moments of all the particles is a constant. Equation (3.1) is the relational *Law of the Conservation of Linear Momentum.*

Let us assume, that of all the gravitational interactions which appear on the right-hand side of equation (2.4), only the nearest neighbors of the ith particle contribute significantly to the interaction. Summing over the nearest neighbors in equation (2.4), we get

$$\sum_{n} m_i \ddot{\mathbf{r}}_i / M(n) - \sum m_j \ddot{\mathbf{r}}_j / M = 0$$

where $M(n)$ is the total mass of the nearest neighbors. Thus, to a good approximation, we can replace the origin of the coordinate system which is the center-of-mass of all the particles in the universe with the origin whose center-of-mass is that of the nearest neighbors.

The weakness of the gravitational interaction is the reason that classical Newtonian mechanics gives such excellent results. It matters very little whether we use the terrestrial laboratory or the stars as the frame of reference. The difference is much too small to be observable. As an example, let us consider the inertial reference frames used in the Special Theory of Relativity. If these coordinate systems are purely mathematical, by which is meant that they are not physical structures, then they are special versions of absolute space. If they are massive, then because of the weakness of the gravitational interaction, they are for all practical purposes equivalent systems of reference in the 'relativistic' sense. But, in theory, if the reference frames are physical structures, then they cannot be equivalent reference frames, since each of the frames are in different gravitational environments.

We conclude that, in practice, there are to a good approximation equivalent reference frames but that, in theory, we can only have equivalent reference frames in the absence of gravitation.

It would appear that the weakness of the gravitational interaction would make it difficult to distinguish Relational Mechanics from Newtonian mechanics. But, as the gravitational field approaches zero, there are traces of Relational Mechanics which are left and cannot be erased from the Newtonian mechanics which emerges in the limit. One such notion is that of inertia. In Newtonian mechanics, it is meaningful to discuss a universe which consists of one particle in motion. Based on that motion, which is relative to absolute space, we can define the inertia of that particle. However, Relational Mechanics, even in the limit of vanishing gravitational interaction, requires that the universe consist of at least two particles and that motion of one particle is discernible only relative to the second particle. Since the gravitational interaction is the source for that motion, the inertia of a particle is a function of the gravitational masses involved. Thus, in equation (2.5), the relative acceleration of the ith particle is multiplied by the relational inertia of that particle, $m_i M(i)/M$. All the masses involved are gravitational masses. In Relational Mechanics there is no distinction between inertial and gravitational masses. In Newtonian mechanics, where the sources for inertial mass and gravitational mass are claimed to be distinct, it is a legitimate question to ask: Why are the two masses equal to each other? No such problem arises in Relational Mechanics.

Accepting the relational interpretation, we can readily establish the connection between relational inertia and Newtonian inertia. In equation (2.5), if $M(i) \gg m_i$, then in practice the Newtonian value for inertia would be the same as its relational value. However, as equation (2.6) illustrates if masses comparable in magnitude are involved, then the relational inertia can differ markedly from the Newtonian. In fact, the relational value is what classical Newtonian mechanics would label the reduced mass of the system.

We have seen that Relational Mechanics does not distinguish between inertial and gravitational mass. In Relational Mechanics, the acceleration of a body in a gravitational field depends on the mass of that body. Also, we established Relational Mechanics as a union between Newtonian mechanics and Mach's principle so that all our conclusions are necessarily that part of Newtonian mechanics which does not rely on an absolute frame of reference. Because of the fundamental nature of mechanics, the changes that constitute what we have called Relational Mechanics will permeate much of physics. We turn, now, to a limited discussion of some of the areas which will be affected.

4. *Hamiltonian Formalism*

We have been discussing Newton's equations of motion and equations (2.4) and (2.5) for the gravitational interactions. We will generalize these equations and indicate how to develop a Hamiltonian formulation which can be used to couple Relational Mechanics to the various other branches of physics. We will not explore the consequences of the relational approach for quantum mechanics, electromagnetic theory, etc., but will leave that material for additional publications.

Let us start with Newton's equations of motion in their most general form.

$$m_i \ddot{\mathbf{r}}_i = \sum_{k \neq i} \mathbf{F}_{ik}; \qquad \mathbf{F}_{ik} = -\mathbf{F}_{ki} \tag{4.1}$$

Proceeding as before, we develop the sequence of equations,

$$m_i m_j(\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_j) = m_j \sum_{k \neq i} \mathbf{F}_{ik} - m_i \sum_{k \neq j} \mathbf{F}_{jk} \tag{4.2}$$

$$\sum_j \frac{m_i m_j}{M}(\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_j) = m_i\left(\ddot{\mathbf{r}}_i - \frac{\sum m_j \ddot{\mathbf{r}}_j}{M}\right) = \sum_{k \neq i} \mathbf{F}_{ik} \tag{4.3}$$

which are generalizations of the equations derived earlier, equations (2.2), (2.4) and (2.5).

As is evident, the statements made about the gravitational field, inertia, coordinate systems, etc., can be taken over verbatim. They are valid in this most general form.

Equation (4.2), on scalar multiplication with $\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j$ and summed over all the particles, becomes

$$\frac{d}{dt}\left[\sum_{i,j} \frac{m_i m_j}{2M}(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)^2\right] = \sum_i \sum_{j \neq i} \mathbf{F}_{ij} \cdot (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) \tag{4.4}$$

The term in brackets is twice the kinetic energy expressed in terms of relative velocities. The repeated summation accounts for the fact that we obtain double the kinetic energy.

The right-hand side of equation (4.4) for conservative fields, i.e., those for which we can write

$$\mathbf{F}_{ij} = -\nabla_{ij} V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$$

reduces to

$$\sum_i \sum_{j \neq i} \mathbf{F}_{ij} \cdot (\mathbf{v}_i - \mathbf{v}_j) = -\sum_i \sum_{j \neq i} \frac{d}{dt} V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \tag{4.5}$$

and determines the potential energy in terms of relative coordinates.

Equations (4.4) and (4.5) give us the Law for the Conservation of Energy expressed in totally relational terms:

$$\frac{d}{dt}\left[\sum_{i,j} \frac{m_i m_j}{2M}(\mathbf{v}_i - \mathbf{v}_j)^2 + \sum_{i,j \neq i} V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)\right] = 0 \tag{4.6}$$

where the term in brackets is equal to twice the total energy.

Having established the relational counterparts to Newtonian kinetic and potential energies, we can develop the relational Lagrangian and Hamiltonian formalisms along lines almost identical to the well-known classical derivations. There is no need to review those procedures.

But a comment is in order here with regards to the variables which occur. The displacement vector $\mathbf{r}_i$ and the velocity vectors $\mathbf{v}_i$ must be considered the independent variables, since the origin of the coordinate system itself is arbitrary. One may be tempted to introduce relative coordinate variables, i.e., $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$ which, of course, could be done, but it leads to complications because there would exist relations among them, for example, $\mathbf{r}_{ij} + \mathbf{r}_{jk} = \mathbf{r}_{ik}$.

To complete the relational description of mechanics, we present the relational form for angular momentum. Take the cross-product of equation (4.2) with $\mathbf{r}_i - \mathbf{r}_j$ and sum over all the pairs of particles. We find

$$\frac{d}{dt}\left[\sum_{i,j} \frac{m_i m_j}{M}(\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j)\right] = \sum_{i,j \neq i} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} \tag{4.7}$$

In words, the result states that the rate of change of the total angular momentum is equal to the total torque impressed. Conservation of angular momentum follows for those situations, for example, conservative fields, for which the right-hand side vanishes. The relational equation is so similar to its Newtonian analog that no additional comment seems necessary. Nonetheless, the result appears less mysterious if it is noted that

$$\sum_{i,j} \frac{m_i m_j}{M}(\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j) = 2\sum_i m_i\left(\mathbf{r}_i - \frac{\sum m_j \mathbf{r}_j}{M}\right) \times \left(\mathbf{v}_i - \frac{\sum m_j \mathbf{v}_j}{M}\right)$$

So again, the classical Newtonian relation appears with the displacement vector and velocity vector of each particle given with reference to the center-of-mass of the universe, or, in physical terms, the position and velocity of each particle is relative to all the particles in the universe.

Discussion

Relational Mechanics is an aspect of Newtonian mechanics which is free from certain of its defects. The notion of absolute space which was a basic source of the dissatisfaction with Newtonian mechanics does not appear in Relational Mechanics. Where Newtonian mechanics requires that inertial mass and gravitational mass be considered distinct concepts, Relational Mechanics does not. Where the gravitational field in Newtonian mechanics is characterized as imparting an absolute acceleration to a body independently of the mass of that body, Relational Mechanics denies the verifiability of such an absolute acceleration and demonstrates that a gravitational field imparts a relative acceleration to a body that does depend upon the mass of the body. For example, Kepler's problem for two such bodies has for its solution the relative displacements, velocities, etc., as the observables for the two-body problem.

Relational Mechanics does provide preferred frames of reference. This raises the question of whether or not an Einsteinian relativistic formulation of Relation Mechanics is possible. The answer is yes. Relational Mechanics is without doubt a classical relativistic theory. However, it fails to include a proper treatment of the relativistic concept of time. But this aspect of the theory can be, and, in fact, has been, developed. The procedure of E. A. Milne (Milne, 1948) answers the problem admirably. Its application to the present theory will be the content of another report.

References

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